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On Stability of Limit Cycles of a Prototype Problem of Piecewise Linear Systems

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Summary. The purpose of this paper is to develop a machinery to analyze existence and stability of limit cycle of a prototype of piecewise linear systems, possibly with delays in switching rules. The study of this type of problems is motivated by modelling cell cycle regulation. The results are applied to a cell cycle model of fission yeast. It is shown that the cell cycle model has a limit cycle and it is stable and criterion of the stability regions are also given.

3.1 Introduction

Consider the following piecewise linear system of prototype

$$\begin{aligned} \dot{x}(t) &= A_\alpha x(t) + B, \quad \text{for } t > t_0, \\ x(s) &= \varphi(s), \quad \text{for } t_0 - \tau \le s \le t_0 \end{aligned} \tag{3.1}$$

with a switching rule

$$\alpha(x) \in \{1, 2, ...\} \tag{3.2}$$

where $x \in \mathbb{R}^n$ is the state, $\varphi(s)$ is given in $\mathbb{R}^n$, A_α is an invertible $n \times n$ matrix and B is an $n \times 1$ matrix.

In order to distinguish the time at which we inspect the state from the variable passing through the interval $[t_0 - \tau, t_0]$ we shall, as usual in the theory of delay equations (see Hale [13]), write throughout the paper $x_t(s) := x(t+s)$ for $t \ge t_0$ and $t_0 - \tau \le s \le t_0$. With this notation, x_t is the state at time t. Clearly the solution to (3.1) is

$$x(t) = \begin{cases} e^{A_\alpha (t-t_0)}(\varphi(t) + A_\alpha^{-1} B) - A_\alpha^{-1} B, & \text{for } t_0 - \tau \le t \le t_0 \\ e^{A_\alpha (t-t_0)}(\varphi(t_0) + A_\alpha^{-1} B) - A_\alpha^{-1} B, & \text{for } t > t_0 \end{cases} \tag{3.3}$$

for a fixed α.

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The motivation of studying this class of piecewise linear systems is highly inspired by a desire for understanding the complexity of cell cycle regulation and for making mathematical analysis accessible to these complex systems, as comprehensive as possible. For a detailed study on how a highly nonlinear complex cell cycle model [1] can be reduced to the piecewise linear system described above we refer to [2, 3]. For references on stability analysis of piecewise linear systems without unstructural delay we refer the reader to e.g. [4, 5].

In this paper we shall give a general analysis of the systems defined by (3.1)-(3.2), and prove that there is a limit cycle in the cell cycle system discussed in [3] and that it is locally stable. The stability regions of the limit cycle will also be discussed.

Without loosing insight in detailed analysis, we assume that $\alpha(x) \in \{i, j\}$ and i, j correspond to the following rule

$$\begin{aligned} i &= \begin{cases} 1 & \text{if } Cx_t \leq \theta_1 \\ 2 & \text{if } Cx_t > \theta_1 \end{cases}, \\ j &= \begin{cases} 1 & \text{if } Cx \leq \theta_2 \\ 2 & \text{if } Cx > \theta_2 \end{cases}, \end{aligned} \tag{3.4}$$

with C being a $1 \times n$ matrix. Denote the hyperplanes $Cx - \theta_1 = 0$ and $Cx - \theta_2 = 0$ by S^I and S^D. Note that the index j in (3.4) indicates a delay of τ for (x, α) passing through the hyperplane S^D. Thus we sometimes say the immediate respectively delayed switching plane. For simplicity let S^I lie to the left of S^D and we only consider, throughout the paper, systems where, if $x(t'') \in S^D$ then $x(t) \notin S^D \cup S^I$ for $t \in]t'', t'' + \tau]$. See motivation in Section 3.5.

The paper is organized as follows. In Section 3.2, we study the existence of limit cycles. Then we turn to analysis on stability, especially Section 3.3 deals with local stability and Section 3.4 stability regions. In Section 3.5, we apply the results in Section 3.3 and Section 3.4 to a reduced cell cycle model proposed in [3]. Finally the paper is concluded by some further comments in Section 3.6.

3.2 Existence of Limit Cycle

Assume that a limit cycle generated by (3.3)-(3.4) passes the switching planes in a clockwise consecutive order according to Figure 3.1, and that the delayed switch of this limit cycle after passing S^D from left to right takes place on the right side of S^D, while the delayed switch after passing S^D from right to left takes place on the left side of S^I. Then a limit cycle solution can be constructed by integrate the subsystems according to the switching rules, (3.3) and (3.4), respectively.

Let $\widetilde{x}_1, \widetilde{x}_2, \widetilde{x}_3, \widetilde{x}_4$ be the intersection points between the trajectory generated by (3.3)-(3.4) and the hyperplanes S^I and S^D as indicated in Figure 3.1.

Let the time taken from $\widetilde{x}_i$ to $\widetilde{x}_{i+1}$ be $t_{i,i+1}$ where $i+4 = i, i = 1, 2, 3, 4$. Then $t_{34} < \tau < \min(t_{23}, t_{34} + t_{41})$ according to the assumptions on delay described earlier.

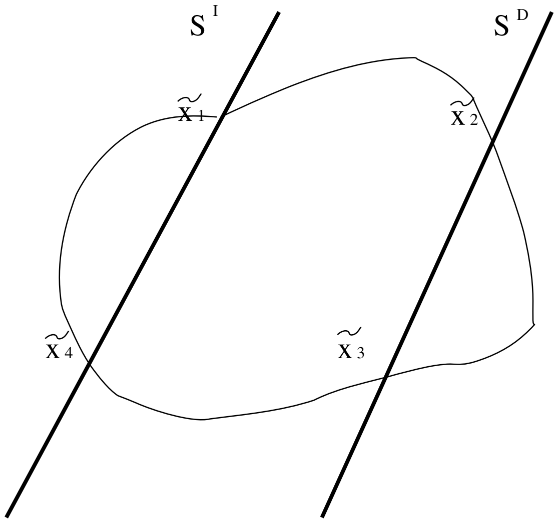

Fig. 3.1. Switching planes and trajectory of limit cycle.

Note that the matrices A_{11}, A_{12}, A_{21} and A_{22} are assumed to be invertible. Thus the solution (3.3), together with the switch rules (3.4), can be written explicitly as follows:

$$\begin{aligned}
\widetilde{x}_2 &= e^{A_{21}t_{12}}(\widetilde{x}_1 + A_{21}^{-1}B) - A_{21}^{-1}B,\\
\widetilde{x}_3 &= e^{A_{22}(t_{23}-\tau)}(e^{A_{21}\tau}(\widetilde{x}_2 + A_{21}^{-1}B) - A_{21}^{-1}B) + A_{22}^{-1}B) - A_{22}^{-1}B,\\
\widetilde{x}_4 &= e^{A_{22}t_{34}}(\widetilde{x}_3 + A_{22}^{-1}B) - A_{22}^{-1}B,\\
\widetilde{x}_1 &= e^{A_{11}(t_{41}-(\tau-t_{34}))}((e^{A_{12}(\tau-t_{34})}(\widetilde{x}_4 + A_{12}^{-1}B) - A_{12}^{-1}B) + A_{11}^{-1}B) - A_{11}^{-1}B.
\end{aligned}$$

Now, by this construction we can in principle formulate the conditions for the existence of limit cycle. Obviously, it takes t_i^* time for actual switch of the system, where $t_1^* = t_{12} + \tau$, $t_2^* = t_{34} + t_{23} - \tau$, $t_3^* = \tau - t_{34}$ and $t_4^* = t_{41} + t_{34} - \tau$.

By successive elimination of $\widetilde{x}_2, \widetilde{x}_3$ in the expression of $\widetilde{x}_1$ we have

$$\begin{aligned}
\widetilde{x}_1 = (I - E_4E_3E_2E_1)^{-1}\,[E_4E_3E_2(E_1 - I)z_1 + E_4E_3(E_2 - I)z_2&\\
+E_4(E_3 - I)z_3 + (E_4 - I)z_4]&
\end{aligned} \tag{3.5}$$

where $E_i = e^{A_i t_i^*}$ and $z_i = A_i^{-1}B$ with $A_1 = A_{21}, A_2 = A_{22}, A_3 = A_{12}, A_4 = A_{11}$. In the same way we obtain

$$\begin{aligned}
\widetilde{x}_4 = (I - E_3E_2E_1E_4)^{-1}\,[E_1E_2E_3(E_4 - I)z_4 + E_1E_2(E_3 - I)z_3&\\
+E_1(E_2 - I)z_2 + (E_1 - I)z_1]\,,&
\end{aligned} \tag{3.6}$$

$$\begin{aligned}
\widetilde{x}_2 = (I - E_{12}E_4E_3E_2E_\tau)^{-1}\,[(E_{12}E_4E_3E_2(E_\tau - I) + E_{12} - I)z_1&\\
+E_{12}E_4E_3(E_2 - I)z_2 + E_{12}E_4(E_3 - I)z_3 + E_{12}(E_4 - I)z_4]\,,&
\end{aligned} \tag{3.7}$$

$$\widetilde{x}_3 = (I - E_{23}E_1E_4E_3E_{34})^{-1}\left[(E_{23}E_1E_4E_3(E_{34} - I) + E_{23} - I)z_2 \right.$$
$$\left. + E_{23}E_1E_4(E_3 - I)z_3 + E_{23}E_1(E_4 - I)z_4 + E_{23}(E_1 - I)z_1\right], \quad (3.8)$$

where $E_\tau = e^{A_1\tau}$, $E_{12} = e^{A_1 t_{12}}$, $E_{23} = e^{A_{21}(t_{23}-\tau)}$ and $E_{34} = e^{A_{21}t_{23}}$. Obviously, $E_{23}E_{34} = E_2$ and $E_\tau E_{12} = E_1$. Since $\widetilde{x}_2$ and $\widetilde{x}_3$ lie on S^D and $\widetilde{x}_1$ and $\widetilde{x}_4$ lie on S^I, respectively, they satisfy $C\tilde{x}_1 - \theta_1 = 0$, $C\tilde{x}_2 - \theta_2 = 0$, $C\tilde{x}_3 - \theta_2 = 0$, $C\tilde{x}_4 - \theta_1 = 0$. Thus we have the following result on the existence of a limit cycle.

Proposition 3.2.1. *Assume that there exists a periodic solution with four switches per cycle and period $t^* = t_1^* + t_2^* + t_3^* + t_4^* > 0$. Assume further that $\tilde{x}_j$:s are defined by (3.5)-(3.8) and $g_1(t_1^*, t_2^*, t_3^*, t_4^*) = C\tilde{x}_1 - \theta_1$, $g_2(t_1^*, t_2^*, t_3^*, t_4^*) = C\tilde{x}_2 - \theta_2$, $g_3(t_1^*, t_2^*, t_3^*, t_4^*) = C\tilde{x}_3 - \theta_2$, $g_4(t_1^*, t_2^*, t_3^*, t_4^*) = C\tilde{x}_4 - \theta_1$. Then the following conditions hold*

$$\begin{cases} g_1(t_1^*, t_2^*, t_3^*, t_4^*) = 0 \\ g_2(t_1^*, t_2^*, t_3^*, t_4^*) = 0 \\ g_3(t_1^*, t_2^*, t_3^*, t_4^*) = 0 \\ g_4(t_1^*, t_2^*, t_3^*, t_4^*) = 0 \end{cases}$$

and the period solution is governed by system with A_{21} on $[0, t_1^*)$, A_{22} on $[t_1^*, t_1^* + t_2^*)$, A_{12} on $[t_1^* + t_2^*, t_1^* + t_2^* + t_3^*)$, and A_{11} on $[t_1^* + t_2^* + t_3^*, t^*)$. Furthermore, the periodic solution is obtained with initial condition $\widetilde{x}_i$ for $i = 1, 2, 3, 4$.

Note that if the initial condition does not belong to any switching surface the existence of a limit cycle still holds, for the trajectory will cross one switching surface after a finite time.

3.3 Local Stability of Limit Cycles

The idea is to analyze the effect of a small perturbation of the initial condition $\widetilde{x}_1$ on S^I that generates a limit cycle (or other points as defined in the previous section) to the first return map. Let the return map be T from some point in a small neighbourhood of $\widetilde{x}_1 \in S^I$, to the point where the trajectory returns to S^I. It is well-known that the limit cycle is locally stable if all eigenvalues of the Jacobian of T are inside the unit circle.

To this end we have to find the Jacobian of the return map. Starting at $x(t_0) = \widetilde{x}_1 \in S^I$, $x(t) = e^{A_{21}(t-t_0)}(x(t_0) + A_{21}^{-1}B) - A_{21}^{-1}B$, if $t < t_{12} + \tau$, thus $\widetilde{x}_2 = e^{A_{21}t_{12}}(\widetilde{x}_1 + A_{21}^{-1}B) - A_{21}^{-1}B$. Now let $x(t_0) = \widetilde{x}_1 + \widetilde{\delta x_1}$ where $\widetilde{\delta x_1}$ is arbitrary and the norm of which is small, but $x(t_0)$ is on the switching plane, i.e. it is such that $C(\widetilde{x}_1 + \widetilde{\delta x_1}) - \theta_1 = 0$. The solution with this initial condition is

$$x(t) = e^{A_{21}t}(\widetilde{x}_1 + \widetilde{\delta x_1} + A_{21}^{-1}B) - A_{21}^{-1}B.$$

Assuming the solution reaches the switching plane S^D at time $t_{12} + \delta_1 t_{12}$ we have

$$x(t_{12}+\delta_1 t_{12}) = e^{A_{21}(t_{12}+\delta_1 t_{12})}(\widetilde{x}_1 + \widetilde{\delta x_1} + A_{21}^{-1}B) - A_{21}^{-1}B,$$

Taylor expanding the term $e^{A_{21}\delta_1 t_{12}}$, together with the fact that $e^{A_{21}t_{12}}(A_{21}\widetilde{x}_1 + B) = A_{21}x(t_{12}) + B$, gives

$$\begin{aligned} x(t_{12}+\delta_1 t_{12}) &= \widetilde{x}_2 + e^{A_{21}t_{12}}\widetilde{\delta x_1} + e^{A_{21}t_{12}}(A_{21}\widetilde{x}_1 + A_{21}^{-1}B)\delta_1 t_{12} + O(\delta_1^2) \\ =& \widetilde{x}_2 + e^{A_{21}t_{12}}\widetilde{\delta x_1} + (A_{21}\widetilde{x}_2 + B)\delta_1 t_{12} + O(\delta_1^2) \end{aligned}$$

Since the trajectory passes S^D at $t_{12}+\delta_1 t_{12}$, $Cx(t_{12}+\delta_1 t_{12}) = \theta_2$. Then neglecting the higher order terms and using $\theta_2 = Cx(t_{12})$ we have

$$Ce^{A_{21}t_{12}}\widetilde{\delta x_1} + Cv_1\delta_1 t_{12} = 0.$$

where $v_1 = A_{21}\widetilde{x}_2 + B$. If $Cv_1 \neq 0$ (that is, the solution is transversal to S^D), then

$$\delta_1 t_{12} = -\frac{Ce^{A_{21}t_{12}}}{Cv_1}\widetilde{\delta x_1}.$$

Now we have

$$x(t_{12}+\delta_1 t_{12}) = \widetilde{x}_2 + W_1\widetilde{\delta x_1}$$

i.e. $\widetilde{\delta x_2} = W_1\widetilde{\delta x_1}$ where $W_1 = \left(I - \frac{v_1 C}{Cv_1}\right)e^{A_{21}t_{12}}$.

Next, let $x(t_0) = \tilde{x}_2 + \widetilde{\delta x_2}$, $\tilde{x}_2 \in S^D$, $\widetilde{\delta x_2}$ is arbitrary and the norm of which is small but $x(t_0) \in S^D$. Compute now solution with this initial condition and assume it reaches the switching plane S^I at time $t_{23}+\delta_2 t_{23}$. By a straightforward calculation as before:

$$\begin{aligned} x(t_{23}+\delta_2 t_{23}) &= \tilde{x}_3 + e^{A_{22}(t_{23}-t_\tau)}e^{A_{21}t_\tau}\widetilde{\delta x_2} \\ &+ A_{22}e^{A_{22}(t_{23}-\tau)}((e^{A_{21}t_\tau}(\tilde{x}_2 + A_{21}^{-1}B) - A_{21}^{-1}B) + A_{22}^{-1}B)\delta_2 t_{23} + o(\delta_2^2) \\ =& \tilde{x}_3(A_{22}\tilde{x}_3 + M)\delta_2 t_{23} + e^{A_{22}(t_{23}-t_\tau)}e^{A_{21}t_\tau}\widetilde{\delta x_2} + o\delta_2^2) \end{aligned}$$

if $t_{34}+\varepsilon_1 < \tau < \min(t_{23}, t_{34}+t_{41}) - \varepsilon_2$ for some $\varepsilon_1, \varepsilon_2 > 0$. Neglecting the higher order terms and use the same argument as that in computing W_1 yields

$$\delta_2 t_{23} = -\frac{Ce^{A_{22}(t_{23}-t_\tau)}e^{A_{21}t_\tau}}{C(A_{22}\tilde{x}_3 + B)}\widetilde{\delta x_2}.$$

Hence

$$x(t_{23}+\delta_2 t_{23}) - \tilde{x}_3 \approx \left(I - \frac{(A_{22}\tilde{x}_3 + B)C}{C(A_{22}\tilde{x}_3 + B)}\right)e^{A_{22}(t_{23}-\tau)}e^{A_{21}\tau}\widetilde{\delta x_2}.$$

Set $W_2 := \left(I - \frac{v_2 C}{Cv_2}\right)$ where $v_2 = A_{22}\widetilde{x}_3 + B$.

Using similar calculations and neglecting the higher order terms, we obtain

$$\widetilde{\delta x_4} = W_3\widetilde{\delta x_3} = W_3W_2W_1\widetilde{\delta x_1}.$$

$$x(t_{41}+\delta_4 t_{41}) - \widetilde{x}_1 = W_4\widetilde{\delta x_4} = W_4W_3W_2W_1\widetilde{\delta x_1}.$$

where

$$W_3 = \left(I - \frac{v_2 C}{C v_2}\right) e^{A_{22} t_{34}},$$
$$W_4 = \left(I - \frac{v_4 C}{C v_4}\right) e^{A_{11} t_4^*} e^{A_{12} t_3^*}$$

and $v_2 = A_{22}\widetilde{x}_3 + B$, $v_4 = A_{11}\widetilde{x}_1 + B$, and $\tilde{x}_2, \tilde{x}_3 \in S^D$, $\tilde{x}_1, \tilde{x}_4 \in S^I$. Note that to derive W_2 and W_4 we needed a technical assumption that there exist small numbers $\varepsilon_1 > 0, \varepsilon_2 > 0$ such that $t_{34} + \varepsilon_1 < \tau < \min(t_{23}, t_{34} + t_{41}) - \varepsilon_2$.

Now the Jacobian of the return map is $W = W_4 W_3 W_2 W_1$. If the eigenvalues of W are inside the unit circle, then the limit cycle under consideration is locally stable. Therefore we have proved the following theorem.

Theorem 3.3.1. *Consider the piecewise linear system* (3.3) *and* (3.4)*. Assume there exists a limit cycle with period* t^* *as described in* Proposition 3.2.1*, and that there exist small numbers* $\varepsilon_1 > 0, \varepsilon_2 > 0$ *such that* $t_{34} + \varepsilon_1 < \tau < \min(t_{23}, t_{34} + t_{41}) - \varepsilon_2$*. Assume further that the limit cycle is transversal to the switching planes* S^D, S^I *at* $\widetilde{x}_1, \widetilde{x}_2, \widetilde{x}_3, \widetilde{x}_4$*, respectively. Then the Jacobian of the return map* T *is given by* $W = W_4 W_3 W_2 W_1$*. Furthermore, the limit cycle (if existing) is locally stable if all eigenvalues of* W *lie inside the unit circle.*

3.4 On Stability Regions

In this section we discuss the question arising from the global analysis of limit cycles. However, the analysis given below applies to both cycles and fixed points. Our analysis leads to a description of a stability region, that is, all points in this region will generate solutions that will converge to either an asymptotically stable fixed point or an asymptotically stable limit cycle.

To find the stability regions we study the maps from a subset of one switching plane to a subset of another switching plane. We will give conditions to ensure that the maps we have found are contractive, which in turn provide the condition for asymptotical stability of fixed points or limit cycles. To find these maps for the delay piecewise linear system (3.3)-(3.4) we have to set up some necessary notations and definitions.

Let S_1, S_2 be two switching planes. Let $x(0) = \widetilde{x}_1 + \Delta_1$. Define t_{Δ_1} as the set of all times $t \geq 0$ such that the trajectory $x(t)$ with initial condition $x(0) \in S_1$ and $x(t)$ in the closure of the solution set on $[0, t]$. Note that we have taken the initial time $t_0 = 0$, which is not restricted. Define also the set of *expected switching times* of the map, called impact map, from Δ_1 in a subset of S_1, $S_1^d - \widetilde{x}_1$ called departure set, that generates the trajectory to Δ_2 in a subset of S_2, $S_2^a - \widetilde{x}_2$ called arrival set, to which the trajectory arrives, as $\mathcal{T} = \{t \mid t \in t_{\Delta_1}, \Delta_1 \in S_1^d - \widetilde{x}_1\}$. We denote $x(t, \widetilde{x})$ the trajectory generated by the initial condition $\widetilde{x}$.

Now we turn to finding such maps, called impact maps, for the system (3.3) and (3.4). Remember that we have four switching possibilities: $S^I \to S^I$, $S^I \to S^D$, $S^D \to S^D$, $S^D \to S^I$, that make four maps: (i) S_d^I to S_a^D from left to right,

(ii) S_d^D to S_a^D from right side of S^D, (iii) S_d^D to S_a^I from right to left, and finally (iv) S_d^I to S_a^I from left side of S^I. Denote also the expected switching time sets as $\mathcal{T}_i$, where i is in accordance with the above four cases.

Let $x(0) = \tilde{x}_1 \in S_d^I$. Then

$$x(t, \tilde{x}_1) = e^{A_{21}t}\tilde{x}_1 + \int_0^t e^{A_{21}(t-s)} B ds. \tag{3.9}$$

We require that $x(t, \tilde{x}_1) \in S_a^D$. Hence t is a switching time. Note that switching time may not be unique.

Let $x_1 = \tilde{x}_1 + \Delta_1 \in S_d^I$, $x_2 = \tilde{x}_2 + \Delta_2 \in S_a^D$, and $\tilde{x}_1 \in S_d^I$, $\tilde{x}_2 \in S_a^D$. Then $C\Delta_1 = C\Delta_2 = 0$. From the expression of $x(t, \tilde{x}_1)$ above, we have

$$\Delta_2 = e^{A_{21}t}\Delta_1 + e^{A_{21}t}\tilde{x}_1 + \int_0^t e^{A_{21}(t-s)} B ds - \tilde{x}_2 = e^{A_{21}t}\Delta_1 + x(t, \tilde{x}_1) - \tilde{x}_2.$$

Since $C\Delta_2 = 0$ and $C\tilde{x}_2 = \theta_2$,

$$Ce^{A_{21}t}\Delta_1 = \theta_2 - Cx(t, \tilde{x}_1).$$

Assume that $Cx(t, \tilde{x}_1) \neq \theta_2$. Then

$$\frac{Ce^{A_{21}t}\Delta_1}{\theta_2 - Cx(t, \tilde{x}_1)} = 1$$

showing that

$$\Delta_2 = e^{A_{21}t}\Delta_1 + (x(t, \tilde{x}_1) - \tilde{x}_2) \cdot 1 = \left(I + \frac{(x(t, \tilde{x}_1) - \tilde{x}_2)C}{\theta_2 - Cx(t, \tilde{x}_1)}\right) e^{A_{21}t}\Delta_1.$$

Therefore

$$H_1(t, \tau) := \left(I + \frac{(x(t, \tilde{x}_1) - \tilde{x}_2)C}{\theta_2 - Cx(t, \tilde{x}_1)}\right) e^{A_{21}t}, \quad \tilde{x}_1 \in S_d^I, \tilde{x}_2 \in S_a^D, t \in \mathcal{T}_1 \tag{3.10}$$

is the desired map from $\Delta_1 \in S_d^I - \tilde{x}_1$ to $\Delta_2 \in S_a^D - \tilde{x}_2$ for all $t \in \mathcal{T}_1$.

In the same manner, we can derive the other three maps, denoted by $H_2(t, \tau)$ from $\Delta_2 \in S_d^D - \tilde{x}_2$ to $\Delta_3 \in S_a^D - \tilde{x}_3$, $H_3(t, \tau)$ from $\Delta_3 \in S_d^D - \tilde{x}_3$ to $\Delta_4 \in S_a^I - \tilde{x}_4$, and $H_4(t, \tau)$ from $\Delta_4 \in S_d^I - \tilde{x}_4$ to $\Delta_5 \in S_a^I - \tilde{x}_5$.

$$H_2(t, \tau) = \left(I + \frac{(x(t, \tilde{x}_2) - \tilde{x}_3)C}{\theta_2 - Cx(t, \tilde{x}_2)}\right) e^{A_{22}(t-\tau)} e^{A_{21}\tau}, \ \tilde{x}_2 \in S_d^D, \tilde{x}_3 \in S_a^D, \forall t \in \mathcal{T}_2 \text{ and } t > \tau \tag{3.11}$$

$$H_3(t, \tau) = \left(I + \frac{(x(t, \tilde{x}_3) - \tilde{x}_4)C}{\theta_1 - Cx(t, \tilde{x}_3)}\right) e^{A_{22}t}, \quad \tilde{x}_3 \in S_d^D, \tilde{x}_4 \in S_a^I, \forall t \in \mathcal{T}_3 \tag{3.12}$$

$$H_4(t, \tau) = \left(I + \frac{(x(t, \tilde{x}_4) - \tilde{x}_5)C}{\theta_1 - Cx(t, \tilde{x}_4)}\right) e^{A_{11}(t-\tilde{t})} e^{A_{12}\tilde{t}},$$
$$\tilde{x}_4 \in S_d^I, \tilde{x}_5 \in S_d^I, \forall t \in \mathcal{T}_4, t - \tilde{t} > 0, \text{ for some } 0 < \tilde{t} < \tau \tag{3.13}$$

where $Cx(t, \tilde{x}_i) \neq \theta_1$, $(i = 3, 4)$ and $Cx(t, \tilde{x}_2) \neq \theta_2$.

We summarize this as a theorem:

Theorem 3.4.1. *Assume that* $Cx(t, \tilde{x}_i) \neq \theta_1$, $(i = 3, 4)$ *and* $Cx(t, \tilde{x}_i) \neq \theta_2$, $(i = 1, 2)$ *for* $\widetilde{x}_1 \in S_d^I$, $\widetilde{x}_2 \in S_d^D$, $\widetilde{x}_3 \in S_d^D$ *and* $\widetilde{x}_4 \in S_d^I$. *Define* H_i *as above,* (3.10)-(3.13). *Then, for any* $\Delta_i \in S_d^i - \tilde{x}_i$ *there exists a* $t \in \mathcal{T}_i$ *such that*

$$\Delta_{i+1} = H_i(t, \tau)\Delta_i,$$

Such $t \in t_{\Delta_i}$ *is the switching time associated with* Δ_{i+1}, *for* $i = 1, 2, 3, 4$, *where* $S^1 = S^4 = S^I, S^2 = S^3 = S^D$.

Furthermore, if the initial states are chosen so that these maps are contractive, then the limit cycle, or fixed point, is stable.

Note also that $C\Delta_i = 0$ in the derivation of H_i. This indicates that the maps H_i, in fact, takes place in $\mathbb{R}^{n-1}$. To see this, let $C^\perp$ be an $n \times (n-1)$ matrix with columns orthonormal to C'. Then $\Delta_{i+1} = H_i(t)\Delta_i$ is equivalent to $C^\perp \tilde{\Delta}_{i+1} = H_i(t, \tau)C^\perp \tilde{\Delta}_i$, where $\tilde{\Delta}_i, \tilde{\Delta}_{i+1} \in \mathbb{R}^{n-1}$. Thus,

$$\tilde{\Delta}_{i+1} = (C^\perp)' H_i(t, \tau) C^\perp \tilde{\Delta}_i.$$

Thus

$$\tilde{\Delta}_{i+1} = \tilde{H}_i(t, \tau)\tilde{\Delta}_i.$$

where $\tilde{H}_i(t, \tau) := (C^\perp)' H_i(t, \tau) C^\perp$.

It is in general not easy to check contraction of these maps. However, if the state space is two-dimensional, then the difficulty is reduced significantly. Note that $H_i(t, \tau)$ becomes scalar. To prove that $H_i(t)$ is contractive is equivalent to proving that $|\tilde{H}_i(t, \tau)| < 1$, for each i, since $\tilde{H}_i$ is a scalar, i.e.

$$|(C^\perp)' H_i(t) C^\perp| < 1. \tag{3.14}$$

Next step is to find the largest interval in S^I and S^D around $\tilde{x}_i$ where the impact map from some $U_i \subset S^I(S^D)$ to the next switch on the switching plane is continuous, and a set of initial conditions of interval in S^I or S^D such that every point in this set has switching time in $\mathcal{T}_i$. Define $C_1(t) = Ce^{A_{21}t}C^\perp$, $C_2(t) = Ce^{A_{22}t}e^{A_{21}\tau}C^\perp$, $C_3(t) = Ce^{A_{12}t}C^\perp$, $C_4(t) = Ce^{A_{11}(t-\tilde{t})}e^{A_{12}\tilde{t}}C^\perp$, $d_1(t) = \theta_2 - Cx(t, \tilde{x}_1)$, $d_2(t) = \theta_2 - Cx(t, \tilde{x}_2)$, $d_3(t) = \theta_1 - Cx(t, \tilde{x}_3)$, and $d_4(t) = \theta_1 - Cx(t, \tilde{x}_4)$. We have

Theorem 3.4.2. *Assume that* (3.14) *holds for all* $t \in \mathcal{T}_i := [t_{i-}, t_{i+}]$. *Define*

$$R_i^C = \min_{t \in \mathcal{T}_i} |\dot{d}_i(t)|/|\dot{C}_i(t)|, \quad \bar{R}_i = \inf_{t \notin \mathcal{T}_i} |d_i(t)|/|C_i(t)|.$$

Then the impact map in the domain $\{\tilde{x}_i + C^\perp \tilde{\Delta}_i : |\tilde{\Delta}_i| < \min\{R_i^C, \bar{R}_i\}\}$ *is a contraction.*

The proof is similar to the ones in [4].

If the piecewise linear system has a local limit cycle with period t^*, and the limit cycle crosses transversely 4 switching planes per cycle we can continue discussing the stability region. To find the stability region of the limit cycle we have to find the conditions for contraction of the four impact maps simultaneously. This is summarized in the following:

Theorem 3.4.3. *Let $\mathcal{T}_i$ be the largest set such that* (3.14) *holds for all $t_i \in \mathcal{T}_i$, $i = 1, 2, 3, 4$. Let $R = \min\{R_i : i = 1, ..., 4\}$. Then the solution starting inside of any of the set defined by $\{\tilde{x}_i + C^{\perp}\tilde{\Delta}_i : |\tilde{\Delta}_i| < R\} \subset S^I$ or S^D, i =1,...,4, converges asymptotically to the limit cycle.*

3.5 Application to a Reduced Cell Cycle Model

The purpose of this section is to illustrate that the study carried out in preceding sections is useful in the global analysis of dynamical behavior of the reduced cell cycle model [3]. The reduced cell cycle system is defined as follows:

$$\dot{x}_{\text{Cdc13t}}(t) = -\, s_1(t-\tau)x_{\text{Cdc13t}}(t) + k_1 M, \tag{3.15}$$

$$\dot{x}_{\text{PreMPF}}(t) = s_2(t)x_{\text{Cdc13t}}(t) - s_3(t, t-\tau)x_{\text{PreMPF}}(t), \tag{3.16}$$

$$y_{\text{MPF}}(t) = x_{\text{Cdc13t}}(t) - x_{\text{PreMPF}}(t), \tag{3.17}$$

$$s_1(t-\tau) = k_2' + s_{slp/ste}(y_{\text{MPF}}(t-\tau)), \tag{3.18}$$

$$s_2(t) = s_{wee}(y_{\text{MPF}}(t)), \tag{3.19}$$

$$s_3(t, t-\tau) = s_{wee}(y_{\text{MPF}}(t)) + s_{25}(y_{\text{MPF}}(t)) + k_2' + s_{slp/ste}(y_{\text{MPF}}(t-\tau)), \tag{3.20}$$

$$s_{25}(z) = \begin{cases} l_{25} & \text{if } z \le \theta_{25/\text{wee}} \\ h_{25} & \text{if } z > \theta_{25/\text{wee}} \end{cases}, \tag{3.21}$$

$$s_{\text{wee}}(z) = \begin{cases} h_{\text{wee}} & \text{if } z \le \theta_{25/\text{wee}} \\ l_{\text{wee}} & \text{if } z > \theta_{25/\text{wee}} \end{cases}, \tag{3.22}$$

$$s_{\text{slp/ste}}(z) = \begin{cases} l_{\text{slp/ste}} & \text{if } z \le \theta_{\text{slp}} \\ h_{\text{slp/ste}} & \text{if } z > \theta_{\text{slp}} \end{cases}, \tag{3.23}$$

where the parameters are

$$\begin{gathered} \tau = 15, \quad k_1 = 0.03, \quad k_2' = 0.03, \quad l_{25} = 0.2, \quad h_{25} = 5, \quad \theta_{25/\text{wee}} = 0.25, \\ h_{\text{wee}} = 1.3, \; l_{\text{wee}} = 0.15, \; l_{\text{slp/ste}} = 0, \; h_{\text{slp/ste}} = 1.3, \; \theta_{\text{slp/ste}} = 0.4, \quad \mu = 0.005. \end{gathered} \tag{3.24}$$

In the original model [1], the cell mass M is a slow time dependent variable. We here treat M as a constant parameter in order to examine model behaviour for different values of M. In the following analysis $M = 1.8$.

Let $x = (x_{\text{Cdc13t}} \;\; x_{\text{PreMPF}})'$ represent the state of the cell cycle system and $u_{ext} = M$ the external input, and let $y = y_{\text{MPF}}$ be the output from the cell cycle system. Then, the DPL described in the preceding section can be put in the matrix form

$$\dot{x} = Ax + B, y = Cx, \tag{3.25}$$

where $C = (1 \;\; -1)$, $A = \left(\begin{smallmatrix} -s_1(t-\tau) & 0 \\ s_2(t) & -s_3(t,t-\tau) \end{smallmatrix}\right)$, and s_1,s_2,s_3 are combinations of step functions defined by (3.18)-(3.20) and $B = (k_1 u_{ext} \;\; 0)'$ and k_1 a constant parameter. The system matrix A takes four possible forms, indexed by A_{ij}, $i, j \in$

$\{1,2\}$, where $i = i(y(t))$ and $j = j(y(t-\tau))$ (A change of index i corresponds to a change of step functions s_{25} and s_{wee} and a change of j to a change of $s_{slp/ste}$). Then

$$\dot{x}(t) = A_{ij}x(t) + B \tag{3.26}$$
$$y(t) = Cx(t), \tag{3.27}$$

where i and j correspond to the following switching rules

$$\begin{aligned} i(y(t)) &= \begin{cases} 1, & \text{if } y(t) \le \theta_{25/wee} \\ 2, & \text{if } y(t) > \theta_{25/wee} \end{cases}, \\ j(y(t-\tau)) &= \begin{cases} 1, & \text{if } y(t-\tau) \le \theta_{slp/ste} \\ 2, & \text{if } y(t-\tau) > \theta_{slp/ste} \end{cases}, \end{aligned} \tag{3.28}$$

and $\theta_{25/wee}$ and θ_{slp} correspond to the switching thresholds of the different step functions. The DPL-model is illustrated in Figure 3.2. The resulting A_{ij}-matrices are obtained from (3.15)-(3.23), and correspond to

$$\begin{aligned} A_{11} &= \begin{bmatrix} -(k_2' + l_{slp/ste}) & 0 \\ h_{wee} & -(h_{wee} + l_{25} + k_2' + l_{slp/ste}) \end{bmatrix} \\ A_{12} &= \begin{bmatrix} -(k_2' + h_{slp/ste}) & 0 \\ h_{wee} & -(h_{wee} + l_{25} + k_2' + h_{slp/ste}) \end{bmatrix} \\ A_{21} &= \begin{bmatrix} -(k_2' + l_{slp/ste}) & 0 \\ l_{wee} & -(l_{wee} + h_{25} + k_2' + l_{slp/ste}) \end{bmatrix} \\ A_{22} &= \begin{bmatrix} -(k_2' + h_{slp/ste}) & 0 \\ l_{wee} & -(l_{wee} + h_{25} + k_2' + h_{slp/ste}) \end{bmatrix}. \end{aligned} \tag{3.29}$$

Here $h_{slp/ste}$, h_{wee}, h_{25} and $l_{slp/ste}$, l_{wee}, l_{25} are the high and low values of the step functions and k_2' is the parameter from the original NT-model [1]. Note that the matrices A_{ij} are invertibe and have all eigenvalues real negative.

Following the discussion in Sections 3.2 and 3.3 we could find a limit cycle going from $\tilde{x}_1 = (1.646, 1.396)$ on S^I: $(1,-1)x = \theta_{25/wee}$, it reaches $\tilde{x}_2 = (1.646, 1.246)$ on S^D: $(1,-1)x = \theta_{slp/ste}$, continues to $\tilde{x}_3 = (0.412, 0.012)$ on S^D, then to $\tilde{x}_4 = (0.257, 0.007)$ on S^I and finally goes back to $\tilde{x}_1$. The period is 112.253 and the switching times $t_1^* = 0.023$, $t_2^* = 16.13$, $t_3^* = 0.40$ and $t_4^* = 95.70$. This is depicted in Figure 3.2.

Note that in order to prove the local stability of a limit cycle we have to show that the Jacobian of the return map W defined in Theorem 3.3.1 should have all eigenvalues inside the unit circle.

If all parameters are fixed as in (3.24), and cell mass $M = 1.8$, we can easily compute the eigenvalues of W to check if the eigenvalues are inside the unit circle. The eigenvalues of W are both ≈ 0.

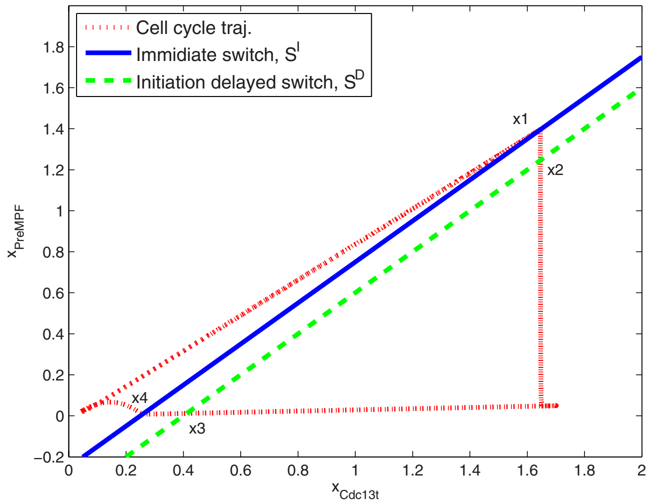

Fig. 3.2. A numerical simulation of the reduced cell cycle model (3.15)-(3.23) together with the switching lines S^I and S^D.

The following estimation will allow us to find parameters such that a limit cycle is locally stable if existing. It is well-known that $|\lambda_i(W)| \leq \|W\|$, where λ_i is denoted as the eigenvalues of W and $\|\cdot\|$ is the norm of an operator $\cdot$ and we shall take the spectral norm, i.e. $\|\cdot\| = \max_{i\in 1,2}(\lambda_i((\cdot)'(\cdot)))^{1/2}$. Now

$$\|W\| \leq \|W_4\|\|W_3\|\|W_2\|\|W_1\|.$$

Then, to guarantee $|\lambda_i(W)| < 1$, it suffices to find conditions such that $\|W_i\| < 1$. To this end we estimate the norms of the matrices W_i.

Lemma 3.5.1. *Let* $A = \begin{bmatrix} \alpha & 0 \\ \gamma & \beta \end{bmatrix}$ *with* $\alpha, \beta, \gamma \in \mathbb{R}$. *If* $\alpha \neq \beta$,

$$e^{2(\alpha+\beta)t} < 1$$

$$1 - (e^{2\alpha t} + e^{2\beta t} + \left(\frac{e^{\alpha t} - e^{\beta t}}{\alpha - \beta}\right)^2 \gamma^2) + e^{2(\alpha+\beta)t} > 0$$

or if $\alpha = \beta$,

$$e^{4\alpha t} < 1$$

$$1 - e^{2\alpha t}(1 + \gamma^2) + e^{4\alpha t} > 0,$$

then $\|e^{At}\| < 1$.

Proof. By a straightforward calculation

$$e^{A't}e^{At} = \begin{bmatrix} e^{2\alpha t} + \left(\frac{e^{\alpha t}-e^{\beta t}}{\alpha-\beta}\right)^2 \gamma^2 & e^{\beta t}\left(\frac{e^{\alpha t}-e^{\beta t}}{\alpha-\beta}\right)\gamma \\ e^{\beta t}\left(\frac{e^{\alpha t}-e^{\beta t}}{\alpha-\beta}\right)\gamma & e^{2\beta t} \end{bmatrix},$$

if $\alpha \neq \beta$. Then the eigenvalues of $e^{A't}e^{At}$ lie inside of the unit circle is equivalent to

$$e^{2(\alpha+\beta)t} < 1$$

$$1 - (e^{2\alpha t} + e^{2\beta t} + \left(\frac{e^{\alpha t} - e^{\beta t}}{\alpha - \beta}\right)^2 \gamma^2) + e^{2(\alpha+\beta)t} > 0.$$

Similarly if $\alpha = \beta$, then

$$e^{A't}e^{At} = e^{2\alpha t}\begin{bmatrix} (1+\gamma^2) & \gamma \\ \gamma & 1 \end{bmatrix},$$

The second alternative follows.

Lemma 3.5.2. *Let* $C = \begin{bmatrix} 1 & -1 \end{bmatrix}$, v_i *defined earlier be* $\begin{bmatrix} a_i \\ b_i \end{bmatrix}$. *Then* $\|W_i\| < 1$, $i = 1, 2, 3, 4$, *if*

$$\sqrt{\frac{2(a_i^2 + b_i^2)}{(a_i - b_i)^2}} < \frac{1}{\|e^{A_i}\|},$$

where $A_1 = A_{21}t_{23}$, $A_2 = A_{22}t_{23} + (A_{21} - A_{22})\tau$, $A_3 = A_{12}t_{34}$, $A_4 = A_{11}t_4^* + A_{12}t_3^*$.

Proof. Since A_{21} and A_{22} commute, and A_{11} and A_{12} commute, we have $e^{A_{22}(t_{23}-\tau)}e^{A_{21}\tau} = e^{A_{22}t_{23}+(A_{21}-A_{22})\tau} = e^{A_2}$ and
$e^{A_{11}t_4^*}e^{A_{12}t_3^*} = e^{A_{11}t_4^*+A_{12}t_3^*} = e^{A_4}$.

A simple calculation yields that the eigenvalues of $(I - \frac{v_iC}{Cv_i})'(I - \frac{v_iC}{Cv_i})$ are 0 and $\frac{2(a_i^2+b_i^2)}{(a_i-b_i)^2} > 0$, where $a_i \neq b_i$ according to the definition of v_i. Then

$$\|I - \frac{v_iC}{Cv_i}\| = \sqrt{\frac{2(a_i^2 + b_i^2)}{(a_i - b_i)^2}}.$$

Hence

$$\|W_i\| \leq \|I - \frac{v_iC}{Cv_i}\|\|e^{A_i}\| < 1,$$

completing the proof.

3.6 Conclusions

We have investigated a class of piecewise linear systems with explicit delay in this paper. The main contribution is giving a set of conditions for local stability of the limit cycle and stability regions of such solutions. Although it is not possible to provide a fully analytical result, our theory provides a computationally checkable tool based on a rigorous analysis. To deal with unstructural delay was new to our best knowledge.

It is worth pointing out that our analysis, with some small modifications, can be carried out for several switching surfaces and also if the delay occurs in a different way. For the essence of the analysis we have chosen the DPL-structure which we think is the most representative (also in the degree of difficulty).

The theory developed in this paper can also be applied to other models, without assuming that the subsystem matrices are invertible or Hurwitz, by a slight modification in our proofs.

Piecewise linear systems with memory delay both in states and switching rules are under investigation. This will hopefully allow us to analyze systems of delay-differential equation such as the one used in [7].

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